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Some Results on Sinai Diffusion

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The diffusion of a particle in a one-dimensional random force field (Sinai diffusion) is studied using the replica method. This method, which maps the problem onto a quantum problem, is shown to be a simple and direct way to calculate the long-time diffusive behavior. Results for the distribution of the local Green's function, the particle distribution, and persistence are obtained.

KEY WORDS: Sinai diffusion; persistence; replica method.

1. INTRODUCTION

The diffusion of a particle in one dimension (1d) in a random quenched force field (Sinai diffusion[1]) has been widely studied by a number of authors. Sinai diffusion is a model for a number of physical phenomena including the dynamics of random field magnets, dislocation dynamics and diffusion of charged particles on one dimensional polymers. These applications are discussed in an excellent and comprehensive review by Bouchaud *et al.*[2] These authors also give an extensive bibliography, present a variety of analytical techniques and make analogies with the directed walk amongst traps and the theory of Levy flights. More recently the problem has been studied by a simple real space renormalization group method by Fisher *et al.*[3] who obtain exact results for the long time or long distance behavior of the particle distribution and some persistent properties.

In this paper we apply the replica method to the Sinai diffusion problem. The replica method was also discussed in the review article of Bouchaud *et al.* from a somewhat different point of view. Here we show that the replica method is a very convenient and simple method to study the long time and large distance behavior. This method maps the problem

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onto a 1d quantum problem which, it turns out, is easily solved and this formulation provides some additional results and insights into this interesting problem. These are the reasons for contributing another paper on this subject. The main results that we obtain are the distribution function of the local Greens function, the particle distribution function and the persistence probability, all in the long time limit. The effect of branching in the lattice on the diffusion behavior is also considered.

2. REPLICA METHOD

Diffusion on an infinite one-dimensional lattice is described by

$$\dot{P}_l = -H_{ll'} P_{l'} \tag{1}$$

where P_l is the occupation probability of site l and hopping occurs between nearest neighbor sites with rates $H_{l,l+1}$ $(l+1 \to l)$ and the reverse $H_{l+1,l}$ and conservation of probability requires $H_{l,l} = -(H_{l+1,l} + H_{l-1,l})$. In the Sinai model

$$\begin{aligned} H_{l+1,l} &= -1 - \Delta_{l+1,l} \\ H_{l,l+1} &= -1 + \Delta_{l+1,l} \end{aligned} \tag{2}$$

where the Δ are independent random variables on each bond with

$$\begin{aligned} \langle \Delta_{l+1,l} \rangle &= \Delta \\ \langle \Delta_{l+1,l}^2 \rangle_c &= \delta \end{aligned} \tag{3}$$

so that Δ is the mean bias (directed to the right or positive l) and δ is the mean square fluctuation in the hopping probability. It is convenient to take the Laplace transform of (1) with the initial condition that the particle starts at the origin:

$$(\varepsilon + H)_{ll'} G_{l'o} = \delta_{l,o} \tag{4}$$

so that G_{lo} is the probability (Laplace transform) of finding the particle at l.

The problem is readily treated by the replica method and we discuss some necessary results. A similar formulation has been used previously.[4] The Hamiltonian H is not Hermitian so we introduce two sets of n component vectors $\vec{x}_l$ and $\vec{y}_l$ on each site and define a generating function

$$Z = \left\langle \int (dx\, dy)\, e^{i\vec{y}(\varepsilon + H).\vec{x} + i\vec{g}.\vec{x} + i\vec{h}.\vec{y}} \right\rangle \tag{5}$$

where the angular brackets indicate an average over the random $\Delta_{l,\, l+1}$ and $\vec{g}_l$ and $\vec{h}_l$ are n component fields on each site. If we integrate out all the sites to the left and right of the origin in (5) the result is

$$Z = \int d\vec{x}_o\, d\vec{y}_o\, s(x_o, y_o) \tag{6}$$

and it is easily shown that (omitting terms of order gh)

$$s(x, y) = \langle \exp[(i\vec{x}.\vec{y} + i\vec{g}_l.G_{lo}.\vec{x} + i\vec{y}.G_{ol}\vec{h}_l)/G_{oo}] \rangle \tag{7}$$

This generating function contains information on the Greens functions of the system. If we set $g = h = o$ we find

$$s_o(\vec{x}.\vec{y}) = \langle e^{i\vec{x}.\vec{y}/G_{oo}} \rangle \tag{8}$$

which allows us to calculate the moments of the diagonal Greens function. It is convenient to let $v = -2i\delta\vec{x}.\vec{y}$ and regard s_o as a function of v. Then

$$\langle G^m_{oo} \rangle = \frac{1}{(m-1)!\,(2\delta)^m} \int_o^\infty v^{m-1}\, dv\, s_o(v) \tag{9}$$

To obtain the off-diagonal Greens function we let $\vec{g}_l = \vec{g}e^{ikl}$, $h = 0$ and expand (7) in g:

$$s = s_o + i\vec{g}.\vec{x}L_k \tag{10}$$

where

$$L_k = \langle G_k/G_{oo} e^{i\vec{x}.\vec{y}/G_{00}} \rangle \tag{11}$$

and G_k in the Fourier transform of G_{lo}. Then

$$\langle G_k \rangle = \frac{1}{2\delta} \int_o^\infty dv\, L_k(v) \tag{12}$$

The persistence[5] $p(t)$ is the probability that the origin (initially unoccupied) is never visited by the diffusing particle during the time t. If we initially place the particle with equal probability $1/N$ on each site of the lattice then $\dot{p}$ is given by the Laplace transform

$$\dot{p}(t) = -\frac{1}{2\pi i N} \int_{-i\infty}^{i\infty} d\varepsilon \left\langle \frac{\sum_l G_{ol}(\varepsilon)}{G_{oo}(\varepsilon)} \right\rangle e^{\varepsilon t} \tag{13}$$

In the Hermitian case $\sum_l G_{ol}(\varepsilon) = 1/\varepsilon$ and (13) reduces to the result given by Stephen and Stinchcombe.[6] To determine the persistence we require the response to the h field in (7) and put

$$s = s_o + i\vec{h}.\vec{g}H \tag{14}$$

where

$$H = \left\langle \frac{\sum_l G_{ol}}{G_{oo}}\, e^{i\vec{x}.\vec{y}} \Big/ G_{oo} \right\rangle \tag{15}$$

so that to determine p we require $H(o)$.

3. DIAGONAL GREENS FUNCTION

An integral equation determining s is easily written down. We first put

$$s(x, y) = e^{-i(\varepsilon\vec{x}.\vec{y} + \vec{g}_o.\vec{x} + \vec{h}_o.\vec{y})} Q_{Lo} Q_{Ro} \tag{16}$$

where L and R indicate the contributions from sites to the left and right of the origin. Then

$$Q_{Ro} e^{-i(\varepsilon\vec{x}.\vec{y} + \vec{g}_o.\vec{x} + \vec{h}_o.\vec{y})} = \int dx'\, dy' \,\langle e^{i(\vec{y} - \vec{y}').(\vec{x} - \vec{x}' + \Delta(\vec{x} + \vec{x}'))}\rangle\, Q_{R1}(x', y') \tag{17}$$

where Q_{R1} comes from sites to the right of site 1. There is a similar equation relating Q_{Lo} to Q_{L-1} but with the sign of Δ reversed. Before proceeding to the solution of (17) it is convenient in place of the parameters Δ, δ and ε of the model and the variable $\vec{x}.\vec{y}$ of (11), to introduce

$$\mu = \Delta/\delta, \qquad E = \varepsilon/\delta^2, \qquad v = -2i\delta\vec{x}.\vec{y} \tag{18}$$

Also if l is distance (site index) on the lattice, t is time and k is wave vector we define the scaled quantities

$$x = \delta l, \qquad \tau = 2\delta^2 t, \qquad q = k/\delta \tag{19}$$

We will be interested in the long time and (or) large distance behavior of the Greens functions and then $E \ll 1$, $q \ll 1$. We can also regard $\delta \ll 1$ and all equations derived below, when expressed in terms of the above scaled variables, will consistently omit terms of higher order in δ. In all equations the replica limit $n = 0$ is taken.

We can reduce (17) to a differential equation if Q is sufficiently slowly varying by the method of steepest descents.[4] We show below that the scale of Q is set by $E^{1/2}$ so that this is justified. Then expanding Q on the right hand side of (17) around $y' = y$ leads to the equation

$$Q_{Ro} e^{-i(\varepsilon\vec{x}.\vec{y} + \vec{g}_o.\vec{x} + \vec{h}_o.\vec{y})} = \left\langle \left[1 + \frac{i}{1+\Delta} \frac{\partial}{\partial \vec{x}} \frac{\partial}{\partial \vec{y}} \cdots \right] Q_{R1} \left(\frac{1+\Delta}{1-\Delta} x, y \right) \right\rangle \quad (20)$$

We first determine $s_o(\vec{x}.\vec{y})$ and set $g = h = o$ and look for a solution of (20) of the form $Q_{Ro} = Q_{R1} = Q_R(\vec{x}.\vec{y})$. Substituting this form in (20), expanding the right hand side in Δ, averaging and introducing the scaled variables (18) we find (after setting the number of replicas $n = o$)

$$\left[(1+v) \frac{d^2}{dv^2} + (1+\mu) \frac{d}{dv} - \frac{E}{4} \right] Q_R = 0 \quad (21)$$

The substitution $z^2 = 1 + v$ reduces (21) to Bessels equation and using the boundary conditions $Q(o) = 1$ and $Q(\infty) = 0$ we find

$$Q_R = \frac{z^{-\mu} K_\mu(E^{1/2} z)}{K_\mu(E^{1/2})} \quad (22)$$

where K is a Bessel function of the second kind and Q_L is obtained by changing the sign of μ. As remarked above the scale is set by $E^{1/2} \ll 1$. The generating function (8) is then (we omit the prefactor in (16))

$$s_o = \frac{K_\mu^2(E^{1/2} z)}{K_\mu^2(E^{1/2})} \quad (23)$$

The average diagonal Greens function from (9) is

$$\langle G_{oo} \rangle = \frac{2^{1-2\mu} \pi\mu}{\delta \sin \pi\mu \Gamma^2(\mu)} \frac{1}{E^{1-\mu}}, \qquad o < \mu < 1$$

$$= \frac{2}{\delta^2 E (\ln E)^2}, \qquad \mu = o \quad (24)$$

For $\mu > 2$ we can expand in E

$$\langle G_{oo} \rangle = \frac{1}{2\delta V} \left(1 - \frac{DE}{V^2} \cdots \right), \qquad \mu > 2 \quad (25)$$

where $V = \mu - 1$, $D = \frac{1}{2}(\mu - 1)/(\mu - 2)$ are the effective bias and hopping probability. For $1 < \mu < 2$ the first term in (25) remains and the second is proportional to $E^{\mu-1}$. These results agree with Bouchaud *et al.*[2] Higher moments of G_{oo} may also be calculated and

$$\langle G_{oo}^m \rangle \sim \frac{1}{E^{m-\mu}}, \qquad m > \mu > o$$
$$\sim \frac{1}{E^m (\ln E)^2}, \qquad \mu = o \tag{26}$$

4. OFF-DIAGONAL GREENS FUNCTION

To determine the off-diagonal Greens function we expand (20) in g (with $h = o$) and look for a solution of the form

$$Q_{Rn} = Q_R + i\vec{g}.\vec{x}e^{ikn} L_{Rk}(\vec{x}.\vec{y}) \tag{27}$$

The same procedure as above leads to the inhomogeneous equation for L_{Rk}

$$v(1+v)\, L''_{Rk} + (1 + (3+\mu)\, v)\, L'_{Rk} + \left(1 + \mu - \frac{iq}{2} - \frac{E}{4} v\right) L_{Rk} = -Q_R/2\delta \tag{28}$$

where the prime indicates differentiation with respect to v. It is useful to note that this problem possesses a Goldstone mode in the sense that applying a uniform field g is equivalent to translation of the y variable in (17) (with $h = 0$). This requires that for $k = 0$

$$L_{Ro} = -\frac{2}{E\delta}\frac{dQ_R}{dv} = \frac{z^{-\mu-1} K_{\mu+1}(E^{1/2} z)}{\delta E^{1/2} K_\mu(E^{1/2})} \tag{29}$$

This result is derived by differentiating (17) with respect to y and using the fact that the kernel depends on $y - y'$ to shift the derivative on the right side. It is easily verified that (29) satisfies (28) (with $q = 0$).

It is convenient to introduce $v = \sinh^2 \theta$ in (28) and write it in the form

$$(\mathscr{H} + 2iq)\, L_{Rk} = \frac{2}{\delta} Q_R \tag{30}$$

where

$$\mathscr{H} = -\frac{d^2}{d\theta^2} - \left(\frac{1}{\sinh\theta \cosh\theta} + (4 + 2\mu)\tanh\theta\right)\frac{d}{d\theta} - 4(1+\mu) + E\sinh^2\theta \tag{31}$$

and we investigate the spectrum of $\mathscr{H}$

$$\mathscr{H}\varphi_n = \lambda_n \varphi_n \tag{32}$$

If we set $E = o$ in (31) it is not difficult to show that for $\mu > 0$ (32) possesses one normalizable bound state $\varphi_o = \cosh^{-2(1+\mu)} \theta$, $\lambda_o = 0$ and a continuum of states for $\lambda > \mu^2$. For $\mu \leqslant 0$ only the continuum survives. The proof of these statements follows by noting that the general solution of (32) when $E = 0$, finite at $\theta = 0$, is a hypergeometric function

$$\begin{aligned} \varphi &= F(\alpha_+, \alpha_-, 1; -\sinh^2 \theta) \\ \alpha_\pm &= 1 + \frac{\mu}{2} \pm \frac{1}{2} \sqrt{\mu^2 - \lambda} \end{aligned} \tag{33}$$

This state is only normalizable if $\lambda = 0$ giving rise to the bound state and leads to a continuum for $\lambda > \mu^2$. Note that $\mathscr{H}$ is not Hermitian, the adjoint states $\phi_n^A = \cosh^{2+2\mu} \theta \phi_n$ and normalization requires

$$\langle \phi_n^A, \phi_n \rangle = \int_o^\infty w(\theta)\, d\theta\, \phi_n^A \phi_n \tag{34}$$

be finite with $w(\theta) = \sinh \theta \cosh \theta$. We parametrize $E = e^{-2\theta_o}$ with $\theta_o \gg 1$ and if we include the $E \sinh^2 \theta$ term in (32) all states become localized and behave as $\exp[-\frac{1}{2} e^{\theta - \theta_o}]$ for $\theta > \theta_o$ i.e., they vanish very rapidly when $\theta > \theta_o$.

The physical significance of λ is the inverse correlation length. Thus for diffusion to the left ($\mu < o$) the probability distribution of the particle will decrease exponentially as $e^{-\mu^2 |x|/2}$. The case $\mu = o$ requires a separate discussion which is given below.

For diffusion to the right ($\mu > o$) the bound state is most important and we need to determine the eigenvalue λ_o and to do this we make use of the Goldstone mode

$$\mathscr{H} L_{Ro} = \frac{2}{\delta} Q_R$$

Suppose L_{Ro} and Q_R are expanded in the eigenstates ϕ_n of $\mathscr{H}$. This leads to

$$\lambda_o = \frac{2}{\delta} \frac{\langle \phi_o^A, Q_R \rangle}{\langle \phi_o^A, L_{Ro} \rangle} \tag{35}$$

The $E \sinh^2 \theta$ term in (31) has little effect on the bound state which is exponentially small for $\theta \sim \theta_o$ so we use $\phi_o^A = 1$ and (22) and (29) to evaluate (35) and find

$$\begin{aligned} \lambda_o &= 2D_\mu E^\mu, & o < \mu < 1 \\ &= \frac{E}{V} - D_\mu E^\mu, & 1 < \mu < 2 \\ &= \frac{E}{V}\left(1 - \frac{ED}{2V^2}\right), & 2 < \mu \end{aligned} \tag{36}$$

where $D_\mu = 2^{1-2\mu} |\Gamma(1-\mu)|/\Gamma(\mu)$, V and D are the effective bias and hopping in (25).

We can now determine the most important part of L_{Rk}. The contribution from the "continuum states" $\lambda > \mu^2$ leads to exponentially decaying terms in the probability distribution of the particle and we omit them and only consider the bound state. This gives

$$L_{Rk} = \frac{2}{\delta} \frac{1}{\lambda_o + 2iq} \frac{\langle \phi_o^A, Q_R \rangle}{\langle \phi_o^A, \phi_o \rangle} \phi_o \tag{37}$$

and when this is substituted in (12)

$$\begin{aligned} \langle G_k \rangle &= \frac{1}{2\delta} \int_o^\infty dv\, L_{Rk} Q_L \\ &= \frac{D_\mu}{\delta^2(\lambda_o + 2iq)}, & \mu < 1 \\ &= \frac{1}{\delta^2 V} \frac{1}{(\lambda_o + 2iq)}, & \mu > 1 \end{aligned} \tag{38}$$

Inverting the Laplace and Fourier transforms we get the probability of finding the diffusing particle at $x \gg o$ and $\tau \gg o$

$$\begin{aligned} \langle G(x, \tau) \rangle &= \frac{\delta}{x} L_\mu^1 \left(\frac{\tau}{2(xD_\mu)^{1/\mu}} \right), & \mu < 1 \\ &= \frac{\delta}{2V} \left(\frac{1}{D_\mu x} \right)^{1/\mu} L_\mu \left(\frac{V\tau - x}{2V} \left(\frac{1}{D_\mu x} \right)^{1/\mu} \right), & 1 < \mu < 2 \\ &= \delta \left(\frac{V}{4\pi D x} \right)^{1/2} e^{-V(x - V\tau)^2}/4Dx, & 2 < \mu \end{aligned} \tag{39}$$

where

$$L^1_\mu = \frac{1}{2\pi i}\int_{-\infty}^{i\infty} dE\, E^{\mu-1} e^{Ez - E^\mu}, \qquad o < \mu < 1$$
$$L_\mu = \frac{1}{2\pi i}\int_{-\infty}^{i\infty} dE\, e^{Ez + E^\mu}, \qquad 1 < \mu < 2 \tag{40}$$

These results exhibit the same scaling properties as those found by Bouchaud *et al.* but there are some differences in detail. In particular for $\mu < 1$ the present result is not identical with the standard Levy flight form given in ref. 2. For $\mu > 2$ the diffusion is normal and a Gaussian distribution with effective bias and diffusion is recovered. For $\mu < 2$ Bouchaud *et al.* have made an analogy with Levy flights.

5. THE CASE $\mu = o$

This case requires a separate discussion as only the "continuum states" contribute. In (30) we put $L_{Rk} = \cosh^{-2}\theta \psi_{Rk}$ where ψ satisfies.

$$\psi''_{Rk} + \frac{1}{\sinh\theta\cosh\theta}\psi'_{Rk} - 2iq\psi_{Rk} = -\frac{2}{\delta}\cosh^2\theta Q \tag{41}$$

where the prime indicates differentiation with respect to θ and for ψ_{Lk} q is replaced by $-q$. We omit the $E\sinh^2\theta$ term which is only important for $\theta \sim \theta_o$ and replace it by the boundary condition that ψ vanishes at $\theta = \theta_o \gg 1$. Considering the region $1 < \theta < \theta_o$ we expand ψ in a Fourier series

$$\psi_R = \sum_{n=o}^{\infty} A_n \cos\omega_n\theta, \qquad \omega_n = (n + \tfrac{1}{2})^\pi/\theta_o \tag{42}$$

and substituting in (41) and neglecting the second term which is exponentially small for $\theta \gg 1$ gives

$$A_n = \frac{4e^{2\theta_o}}{\delta\theta_o^2}\frac{(-1)^n\,\omega_n}{(\omega_n^2 + 2iq)(4 + \omega_n^2)^2} \tag{43}$$

When these results are substituted in (12) we find

$$\langle G_k\rangle = \frac{4e^{2\theta_o}}{\delta^2\theta_o^3}\sum_{n=o}^{\infty}\frac{(-)^n}{\omega_n}\frac{1}{(4+\omega_n^2)^2}\left(\frac{1}{\omega_n^2 + 2iq} + c.c\right) \tag{44}$$

Inverting the Laplace and Fourier transforms gives

$$\langle G(x,\tau)\rangle = \frac{16\delta}{(\ln \tau)^3} \sum_{n=o}^{\infty} \frac{(-)^n}{\bar{\omega}_n} \frac{1}{(4+\bar{\omega}_n^2)^2} e^{-|x|\,\bar{\omega}_n^2/2} \tag{45}$$

where $\bar{\omega}_n = (2n+1)\,\pi/\ln \tau$. These results agree with exact result of Kesten.[(2)]

6. PERSISTENCE

In this case we expand (17) in h (with $g = o$) and look for a solution of the form

$$Q_{Rn} = Q_R + i\vec{h}.\vec{y} H_R(\vec{x}.\vec{y}) \tag{46}$$

This leads to the equation for H_R

$$\mathscr{H}^A H_R = \frac{2}{\delta} Q_R \tag{47}$$

where $\mathscr{H}^A$ is the operator adjoint to (31) (with $\mu \to -\mu$) i.e.,

$$\mathscr{H}^A = \frac{d^2}{d\theta^2} - \frac{1+2\mu \sinh^2 \theta}{\sinh \theta \cosh \theta} \frac{d}{d\theta} - E \sinh^2 \theta \tag{48}$$

For $\mu > o$ the particle comes from the left so we only need consider H_L and we solve for this as above and find

$$H_L = \frac{2}{\delta \lambda_o} \frac{\langle \phi_o, Q_L \rangle}{\langle \phi_o, \phi_o^A \rangle} = \frac{2}{\delta \lambda_o} \tag{49}$$

Using the form of λ_o in (36) we can evaluate the Laplace transform in (13) and find

$$\begin{aligned} p(t) &= 1 - \frac{V\tau}{N\delta}, \qquad \mu > 1 \\ &= 1 - \frac{\tau^{2-\mu}}{N\delta\, D_\mu 2^\mu \Gamma(1+\mu)}, \qquad \mu < 1 \end{aligned} \tag{50}$$

For $\mu = o$ we replace the $E \sinh^2 \theta$ term in (48) by the boundary condition that H vanish at $\theta = \theta_o$. Then (47) can be integrated directly and gives

$$H = (\ln E)^2/6\delta \tag{51}$$

and the persistence

$$p(t) = 1 - \frac{1}{3N\delta}(\ln \tau)^2, \qquad \mu = o \tag{52}$$

The probabilities (50) and (52) give the behavior of a single particle. If we consider an ensemble of ρN independent particles diffusing on independent lattices it is permissible to exponentiate these results leading to for example.

$$p(t) = e^{-\rho(\ln \tau)^2/3\delta}, \qquad \mu = o \tag{53}$$

For a finite density ρ of particles on a given lattice it is not clear that this result is correct.

Finally we note that another disordered diffusion problem in which the persistence has an interesting power law is that considered by Alexander *et al.*[8] In this model the nearest neighbor hopping is symmetric and diverges at small values. Specifically the distribution of $H_{l,\,l+1}$ is $P(H) = (1-\alpha)\, H^{-\alpha}$, $o \leqslant H \leqslant 1$ with $o < \alpha < 1$. In this case using results given in ref. 4 the persistence

$$p(t) = 1 - \frac{C_\alpha}{N}\, t^{(1-\alpha)/(2-\alpha)} \tag{54}$$

where C_α is a constant. For $\alpha > o$ the diffusion is anomalously slow.

7. EFFECT OF BRANCHING

The restriction that the diffusion is in one dimension is clearly crucial as the particle is readily trapped in places where the potential is favorable and this gives rise to the very slow diffusive behavior. It is therefore of interest to investigate what happens if branching is allowed and we briefly consider the diffusive behavior with no bias on a Bethe lattice of coordination number $m+1$ (1d corresponds to $m=1$). The basic integral Eq. (17) is now replaced by

$$Qe^{-i\varepsilon \vec{x}.\vec{y}/m} = \int dx'\, dy' \,\langle e^{i(\vec{y}-\vec{y}').(\vec{x}-\vec{x}'+\Delta(\vec{x}+\vec{x}'))} \rangle\, Q^m(\vec{x}'\vec{y}') \tag{55}$$

Using the steepest descents method we can again pass to a differential equation. Put $Q = e^{-f}$ and f satisfies

$$f'' - (f')^2 - \frac{1}{\sinh\theta \cosh\theta} f' + \omega^2 f + E \sinh^2\theta = o \tag{56}$$

where the prime indicates differentiation with respect to θ and $\omega^2 = 2/\delta(m-1)$. The method requires $m-1 \ll 1$. We want a solution of (56) with $f(o)=o$ and $f(\infty)=\infty$. Such a solution exists for $E=o$ with $f=\frac{1}{4}\omega^2\theta^2+\omega\theta/\sqrt{2}$. This solution gives

$$\langle G_{oo} \rangle \sim \frac{1}{\omega} e^{2/\omega^2} \tag{57}$$

The density of states is then finite at $E=o$ and the diffusive behavior is expected to be normal.

8. CONCLUSIONS

We have studied the diffusion of a particle in 1d in a random quenched force field, a problem first considered by Sinai. The replica method maps the problem in the long time limit onto an easily solved quantum problem. This in turn provides further insight into the model and allows the simple determination of average Greens functions, the particle distribution function and certain persistence properties. In a branched lattice the diffusion is normal.

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